EFFECT OF BUOYANCY FORCES ON A BOUNDARY LAYER NEAR A FREE BOUNDARY

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UDC 532.526

Formal asymptotic expansions of homogeneous liquid flow with free boundaries and vanishing viscosity were constructed in [1]. The main asymptotic flow terms of an inhomogeneous incompressible flow are developed in the present study, and it is shown that the effect of buoyancy forces leads to perturbation transfer to the top of the flow and the presence of velocity profile oscillations in the boundary layer near the free boundary.

For the Navier-Stokes equation with large Reynolds numbers we consider the planar stationary problem of liquid motion in a region D, confined by a free surface Γ and an impenetrable wall S. It is assumed that the liquid is incompressible, stratified in density, and that there is no diffusion. For the model under consideration the equations of motion are [2]

$$\rho(\mathbf{v},\nabla)\mathbf{v} = -\nabla p + \varepsilon^2\Delta\mathbf{v} - \rho\cdot\mathbf{e}_z\cdot F^{-1},\quad \mathbf{v}\cdot\nabla\rho = 0,\quad \operatorname{div}\mathbf{v} = 0,$$
$$p\cdot\mathbf{n} - 2\varepsilon^2\Pi\cdot\mathbf{n} = \mathbf{T},\quad \mathbf{v}\cdot\nabla f_1 = 0,\quad (x,z)\in\Gamma,\quad \mathbf{v}|_s = 0. \tag{1}$$

If the region D is not confined in the x-coordinate, it is assumed that a periodicity condition is assigned, or the behavior of the velocity field is given for $|x| \to \infty$. All quantities in Eq. (1) are dimensionless: The dynamic viscosity coefficient μ is assumed constant, $\varepsilon^2 = 1/\mathrm{Re}$ is a small parameter, $\mathrm{Re} = U_1 L\rho_* \mu^{-1}$ is the Reynolds number, $F = U_1^2/(gL)$ is the Froude number, g is the acceleration due to gravity, U_1, L, ρ_* are the characteristic sizes of velocity, length, and density, the z axis is directed vertically upward, $\mathbf{e}_z = (0, 1)$ is a unit vector along the z axis, **n** is the unit vector of the normal to the free boundary, $f_1(x, z) = 0$ is the equation of Γ in implicit form, Π is the tensor of deformation velocities, $\mathbf{T} = (T_1, T_2)$ is a given load on Γ, while $T_1 = \mathbf{T}\cdot\mathbf{n} = 0$, which corresponds to absence of normal stresses on Γ. It is assumed that $T_2 = O(\varepsilon^2)$, where T_2 is the tangential stress on Γ. We introduce the stream function $\psi(v_x = \partial\psi/\partial z,\ v_z = -\partial\psi/\partial x)$. We note that in the case of finite depth the problem under consideration is that of periodicity functions in the x coordinate for **v**, p, **T**.

Asymptotic expansion solutions of problem (1) for vanishing viscosity $\varepsilon \to 0$ are constructed in the form

$$\psi \sim \sum_{k=0}^{N}\varepsilon^k(\psi_k + \Psi_k + \Phi_k),\quad \rho \sim \sum_{k=0}^{N}\varepsilon^k(\rho_k + R_k + \tilde{r}_k),\quad \zeta \sim \sum_{k=1}^{N}\varepsilon^k\zeta_k, \tag{2}$$

where $z = \zeta(x)$ is the free boundary equation. A similar series is constructed for the function p with coefficients p_k, q_k, χ_k. Boundary layers are formed for vanishing viscosity near the region boundaries. We denote by D_S and D_Γ the boundary layer regions, respectively, near the solid boundary S and the free boundary Γ. Then Ψ_k, $R_,$, q_k are functions of the solution type of the boundary value problem in D_Γ while Φ_k, $\tilde{r}_k$, χ_k are those in the D_S. The functions ψ_k, ρ_k, p_k determine the solution outside D_Γ. It is further assumed that in the region D_Γ the Froude number is of order $O(\varepsilon^2)$, i.e., $F = \lambda\varepsilon^2, \lambda = O(1)$. In this case the action of buoyancy forces is manifested in D_Γ. Everywhere outside the region D_Γ the number F acquires the finite value F_0. We note that the values $F = O(\varepsilon^2)$ in D_Γ correspond to small velocity values near the free surface. Such cases are encountered in the equatorial zone of the ocean [3], where at a depth near the thermocline there occur powerful "subsurface" flows, oriented eastward along the equator, while inverse flows are formed near the ocean surface with substantially lower velocities, oriented westward in the wind direction. Besides, flow drag near the free surface and formation of an opposite flow can occur for spatial nonequilibrium of the tangential stress [4].

The main asymptotic terms ψ_0, ρ_0 are found by solving the problem of an inviscid liquid flow in the region D_0 with free boundary Γ_0. The functions ψ_0, ρ_0 satisfy the system

Rostov-on-Don. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 81-86, May-June, 1985. Original article submitted February 28, 1984.

 0021-8944/85/2603-0374$09.50

$$\rho_0 J(\Delta\psi_0, \psi_0) + \frac{\partial\rho_0}{\partial z} J\left(\frac{\partial\psi_0}{\partial z}, \psi_0\right) + \frac{\partial\rho_0}{\partial x} J\left(\frac{\partial\psi_0}{\partial x}, \psi_0\right) = F_0^{-1}\frac{\partial\rho_0}{\partial x}, J(\rho_0, \psi_0) = 0 \tag{3}$$

with boundary conditions

$$p_0 = \psi_0 = 0, \ (x, z) \in \Gamma_0, \ \mathbf{v}_0 \cdot \mathbf{n}_1|_s = 0$$

and corresponding conditions in x. Here $J(A, B) = (\partial A/\partial x)(\partial B/\partial z) - (\partial A/\partial z)(\partial B/\partial x)$; $\mathbf{n}_1$ is the normal vector to S. It follows from Eq. (3) that $\rho_0 = f(\psi_0)$, where $f(\psi_0)$ is an arbitrary function of ψ_0. It is further assumed that the solution of problem (3) is known.

The functions ψ_k, ρ_k ($k \geqslant 1$) are constructed by means of the first iteration process [5], and satisfy the linear system

$$\sum_{i+j+n=k}\left[\rho_i J(\Delta\psi_j, \psi_n) + \frac{\partial\rho_i}{\partial z} J\left(\frac{\partial\psi_j}{\partial z}, \psi_n\right) + \frac{\partial\rho_i}{\partial x} J\left(\frac{\partial\psi_j}{\partial x}, \psi_n\right)\right] = \tag{4}$$

$$= F_0^{-1}\frac{\partial\rho_k}{\partial x} + \Delta^2\psi_{k-2},$$

$$\sum_{i+j=k} J(\rho_i, \psi_j) = 0, \quad \mathbf{v}_k \cdot \mathbf{n}_1|_S = 0.$$

It follows from Eq. (4) that $\rho_1 = \psi_1 f'(\psi_0)$, $\rho_2 = \psi_2 f'(\psi_0) + 0.5\psi_1^2 f''(\psi_0)$, etc. The boundary conditions at the free boundary will be given below for the system (4).

We now determine the functions Ψ_k, R_k, which are lumped near the free boundary and compensate the inviscid term, generated when the dynamic condition is satisfied for the tangential stress on Γ. Near Γ we introduce local coordinates (r, φ) by the equations

$$x = X(\varphi) + rn_{x0}, \quad z = Z(\varphi) + rn_{z0},$$

where r is the distance between the point (x, z) and Γ_0, the free boundary of the inviscid flow (3), n_{x0}, n_{z0} are the components of the unit vector normal to Γ_0; $x = X(\varphi)$ and $z = Z(\varphi)$ is the parametric equation of the contour Γ_0. Equations for Ψ_k, R_k are found by applying the second iteration of the process of [5] to system (1). Substituting (2) into (1), the equations obtained are written in the local coordinates, and we expand the known coefficients in Taylor series in powers of r. We assume $r = \varepsilon s$, $F = \lambda\varepsilon^2$ for the functions Ψ_k, R_k, and $F = F_0$ for ψ_k, ρ_k. Equating to zero the coefficients of ε^0, ε, ..., ε^N, we derive equations for the determination of Ψ_k, R_k. The function Ψ_0 satisfies a homogeneous boundary-value problem, whose solution is selected in the form $\Psi_0 = 0$. Similarly, $\Psi_1 = 0$. The function Ψ_k ($k \geqslant 2$) satisfies the problem

$$sa(\varphi)\frac{\partial^2\Psi_k}{\partial s^2} + b(\varphi)\frac{\partial^2\Psi_k}{\partial s\partial\varphi} - a(\varphi)\frac{\partial\Psi_k}{\partial s} = \frac{1}{\rho_0|_\Gamma}\frac{\partial^3\Psi_k}{\partial s^3} - \frac{f'(0)}{\lambda\rho_0|_\Gamma} n_{x0}\Psi_k + M_k, \tag{5}$$

$$\left.\frac{\partial^2\Psi_k}{\partial s^2}\right|_{s=0} = \left[\delta^{-2}\frac{\partial^2\psi_{k-2}}{\partial\varphi^2} - \frac{\partial^2\psi_{k-2}}{\partial r^2} - \varkappa\frac{\partial\psi_{k-2}}{\partial r}\right]_{r=0} + N_k, \quad \Psi_k|_{s=\zeta_1} = Q_k,$$

$$\Psi_k|_{s=-\infty} = 0.$$

The coefficients M_k, N_k, Q_k are known, and are not given due to their awkward shape, while $M_2 = Q_2 = 0$, $N_2 = T_2(\varphi)$. Here

$$a(\varphi) = \frac{\partial}{\partial r}[\mathbf{v}_0 \cdot \nabla r]_{r=0}; \quad b(\varphi) = \mathbf{v}_0 \cdot \nabla\varphi|_{r=0},$$

and $\varkappa$ and δ are the curvature and Lamé coefficient of the contour Γ_0. The boundary condition $\Psi_k|_{s=\zeta_1} = Q_k$ is obtained by applying the second iteration of the process to the kinematic condition on Γ.

Consider the case in which the free boundary of the inviscid flow is rectangular ($z = 0$). Now $n_{x0} = 0$, $n_{z0} = 1$. Applying the second iteration of the process to (1), we derive a boundary condition for Ψ_k

$$sa(x)\frac{\partial^3\Psi_k}{\partial s^3} + b(x)\frac{\partial^3\Psi_k}{\partial x\partial s^2} = \frac{1}{\rho_0|_\Gamma}\frac{\partial^4\Psi_k}{\partial s^4} + \beta\frac{\partial\Psi_k}{\partial x} + \widetilde{M}_k, \tag{6}$$

$$\left.\frac{\partial^2\Psi_k}{\partial s^2}\right|_{s=0} = \frac{\partial^2\psi_{k-2}}{\partial z^2} - \frac{\partial^2\psi_{k-2}}{\partial x^2} + \widetilde{N}_k, \quad \Psi_k|_{s=\zeta_1} = \widetilde{Q}_k, \quad \Psi_k = \frac{\partial\Psi_k}{\partial s} = 0 \ (s = -\infty),$$

where $\beta = \lambda^{-1}f'(0)/\rho_0|_\Gamma$; $f'(0) = (\partial\rho_0/\partial z)/v_{x0}|_{z=0}$; $\tilde{M}_2 = \tilde{Q}_2 = 0$, $\tilde{N}_2 = T_2(x)$; $a(x) = [\partial v_{z0}/\partial z]_{z=0}$; $b(x) = v_{x0}|_{z=0}$; $s = z/\varepsilon$.

We determine the boundary conditions for the system (4) on the free boundary Γ. We write the equation for Γ in the form $r = \zeta(\varphi) = \varepsilon\zeta_1(\varphi) + \varepsilon^2\zeta_2(\varphi) + \ldots$ (here it has been taken into account that $r = 0$ is the equation of Γ at $\varepsilon = 0$). Applying simultaneously the first and second iteration processes to the kinematic condition, the dynamic condition for the normal stress, and taking into account the condition $\Psi_k = Q_k(s = \zeta_1)$, we derive the relations

$$\zeta_k \frac{\partial\psi_0}{\partial r}\bigg|_{r=0} + \psi_k|_{r=0} = E_k,\ \ p_k + q_k + \zeta_k\frac{\partial p_0}{\partial r} + \frac{2}{\delta}\frac{\partial^2\psi_{k-2}}{\partial r\partial\varphi} - \frac{2\varkappa}{\delta}\frac{\partial\psi_{k-2}}{\partial\varphi} = \tilde{G}_k\ (r=0),$$

where $k \geqslant 1$; $E_1 = \tilde{G}_1 = \psi_{-1} = 0$.

The boundary layer functions Φ_k, r_k, χ_k occur in the region D_S and compensate the inviscid terms, generated when the sticking conditions in (1) are satisfied. The function Φ_0 satisfies a homogeneous boundary condition, whose solution is written in the form $\Phi_0 = 0$. In the case $F = O(1)$, we obtain is D_S for Φ_1 a nonlinear problem, which, as in [1], reduces to the Prandtl boundary layer equations of a homogeneous liquid. Let $F = \lambda_1\varepsilon$ in D_S. To determine Φ_1 we now apply the boundary layer method in the same way as in deriving Eq. (5). We introduce (r_1, φ_1), the local orthogonal coordinates near the boundaries of S. The problem for $\Phi_1(\xi, \varphi_1)$ is

$$\frac{1}{\delta_1}\left(\frac{\partial\Phi_1}{\partial\xi} + b_1\right)\frac{\partial^2\Phi_1}{\partial\xi\partial\varphi_1} + \left(-\frac{1}{\delta_1}\frac{\partial\Phi_1}{\partial\varphi_1} + \delta_1 a_1\right)\frac{\partial^2\Phi_1}{\partial\xi^2} - a_1\frac{\partial\Phi_1}{\partial\xi} = \frac{\partial^3\Phi_1}{\partial\xi^3} - \frac{f'(0)}{\lambda_1\rho_0|_\Gamma}n_{x1}\Phi_1,$$

$$\Phi_1|_{\xi=0} = 0,\ \ \frac{\partial\Phi_1}{\partial\xi}\bigg|_{\xi=0} = -\frac{\partial\psi_0}{\partial r_1}\bigg|_{r_1=0},\ \ \Phi_1|_{\xi=\infty} = 0,$$

where $\xi = r_1/\varepsilon$; $b_1(\varphi_1) = \partial\psi_0/\partial r_1$ $(r_1 = 0)$; $a_1 = -\delta_1^{-1}\partial^2\psi_0/\partial r_1\partial\varphi_1(r_1 = 0)$; δ_1 is the Lamé coefficient of the contour S, $\mathbf{n}_1 = (n_{x1}, n_{z1})$.

Consider the case of a rectilinear boundary S: $z = -h$. The action of buoyancy forces is now manifested in D_S for $F = \lambda\varepsilon^2$. The boundary layer equation for Φ_1 is obtained in the form

$$\left(\frac{\partial\Phi_1}{\partial\xi_1} + v_{x0}|_S\right)\frac{\partial^3\Phi_1}{\partial x\partial\xi_1^2} + \left(-\frac{\partial\Phi_1}{\partial x} + \xi\frac{\partial v_{z0}}{\partial z}\bigg|_S\right)\frac{\partial^3\Phi_1}{\partial\xi_1^3} = \frac{1}{\rho_0|_S}\frac{\partial^4\Phi_1}{\partial\xi_1^4} + \beta_1\frac{\partial\Phi_1}{\partial x},$$

$$\Phi_1 = 0, \frac{\partial\Phi_1}{\partial\xi_1} = -v_{x0}\ \ (z = -h);\ \ \Phi_1 = \frac{\partial\Phi_1}{\partial\xi_1} = 0\ \ (\xi_1 = \infty),$$

where $\beta_1 = \lambda^{-1}f'(0)/\rho_0|_S$; $\xi_1 = (z + h)/\varepsilon$.

Thus, to integrate the system (1) we initially determine the flow of an inviscid liquid (3), then the flow in the boundary layers D_S and the first approximation (4), and then the flow in the boundary layer near D_Γ. In the absence of a solid boundary the flow in the boundary layer near Γ is determined after integration of the system (3).

Example. Let the flow of the inviscid liquid be given by the velocity field $v_{x0} = U(z)$, $v_{z0} = 0$ and the density distribution $\rho_0 = \rho_0(z)$. The region D_0 is the half-space $z \leqslant 0$, and the free boundary Γ_0 is rectilinear. The tangential stress on Γ is given by the equation $T_2(x) = -U'(0) + T_* e^{-\omega x}$ for $x > 0$ and $T_2(x) = -U'(0)$ for $x < 0 (\omega > 0)$. The boundary layer problem near Γ acquires the form

$$\frac{\partial^4\Psi_2}{\partial s^4} + \beta\frac{\partial\Psi_2}{\partial x} - U(0)\frac{\partial^3\Psi_2}{\partial x\partial s^2} = 0, \tag{7}$$

$$\Psi_2|_{s=0} = 0,\ \ \frac{\partial^2\Psi_2}{\partial s^2}\bigg|_{s=0} = T_* e^{-\omega x}\ \ (x > 0),\ \frac{\partial^2\Psi_2}{\partial s^2}\bigg|_{s=0} = 0\ \ (x < 0),$$

$$\Psi_2 = \frac{\partial\Psi_2}{\partial s} = 0\ \ (s = -\infty).$$

It was here taken into account that $\zeta_1 = 0$, $\rho_0(0) = 1$, $\beta = \lambda^{-1}\rho_0'(0)/U(0)$. It is further assumed that a stable density stratification is given $\rho_0'(0) < 0$ [2] and $U(0) > 0$; therefore $\beta < 0$. The problem (7) is solved by a Fourier transformation in x. The solution Ψ_2 is written as

$$\Psi_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\widetilde{T}(\sigma)}{\nu_1^2 - \nu_2^2} \left(e^{-\nu_1 s} - e^{-\nu_2 s}\right) e^{-i\sigma x} d\sigma, \tag{8}$$

where $\tilde{T}(\sigma) = (\omega - i\sigma)^{-1}(2\pi)^{-1/2}$, ν_1 and ν_2 satisfy the equation $\nu^4 + i\sigma U(0)\nu^2 - i\sigma\beta = 0$ and have the form

$$\nu_{1,2} = \sqrt{\frac{U(0)}{2}} \sqrt{-i\sigma \pm \sqrt{-\sigma^2 + \frac{4i\sigma\beta}{U^2(0)}}},$$

while those root branches are chosen for which $\nu_1 < 0$ and Real $\nu_2 < 0$. The integral (8) is calculated by means of the residue theory, for $x > 0$ the integration region G is selected in the lower half-plane Real $\sigma \leqslant 0$, confined by a semicircle of large radius R with center at the origin of coordinates, a semicircle of infinitely small radius δ_0 with the same center, and the real axis segments $\sigma(-R, -\delta_0)$, (δ_0, R). Continuing analytically the integrand function in $\bar{G}$, applying the Cauchy residue theorem, the Jordan lemma, and carrying out the tendendies $R \to \infty$, $\delta_0 \to 0$, we find

$$\Psi_2 = -\frac{2T_* e^{-\alpha s} \sin \gamma s}{U(0)\sqrt{\omega}\sqrt{\sigma_0 - \omega}} e^{-\omega x} \quad (x > 0), \tag{9}$$

where $\alpha = -\frac{1}{2} \sqrt{U(0)}\sqrt{\sqrt{\sigma_0\omega} - \omega}$; $\gamma = \frac{1}{2}\sqrt{U(0)}\sqrt{\omega + \sqrt{\sigma_0\omega}}$; $\sigma_0 = 4|\beta|U^{-2}(0)$ $(\sigma_0 > \omega)$. For $x < 0$ we also apply the Cauchy theorem, taking into account that $\sigma = 0$ is a branching point. We introduce the integration region G_1 in the upper half-plane Real $\sigma \geqslant 0$, confined by semicircles of radii R and δ_0 with a cut along the imaginary semiaxis from the point $\sigma = 0$. As a result of integration over G and the transformation to the limit at $R \to \infty$, $\delta_0 \to 0$ we obtain

$$\Psi_2 = -\frac{1}{\pi U(0)} \int_0^{\infty} \frac{\sin\left(\sqrt{\frac{U(0)}{2}}\sqrt{\sqrt{\xi(\sigma_0+\xi)} - \xi}\right)s}{U(0)(\omega+\xi)\sqrt{\xi(\sigma_0+\xi)}} e^{\xi x} d\xi \quad (x < 0). \tag{10}$$

For $x \to -\infty$ and fixed s the integral (10) has the asymptotic representation [6]

$$\Psi_2 = -\frac{1.226}{\pi\omega\sigma_0^{1/4}\sqrt{2U(0)}} \frac{s}{(-x)^{3/4}} + o\left(\frac{1}{(-x)^{3/4}}\right).$$

It follows from Eqs. (9), (10) that the velocity profile in the boundary layer oscillates along the z-coordinate.

We note that if $F = O(1)$ in the boundary layer region D_Γ, in Eq. (7) one must put $\beta = 0$. Equation (7) is easily solved by means of the Laplace transform. The function $\Psi_2(x, s)$ is now monotonic in s, and $\Psi_2 = 0$ for $x < 0$, i.e., perturbations are not transferred from above the flow. Thus, the action of buoyancy forces leads to perturbation transfer from above the flow and the presence of oscillations over the z-coordinate in the velocity profile at the boundary layer near the free boundary.

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